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Ph. Sem - III
paper - XII, unit-1st
classical mechanics

Generalised potential

If in place of ordinary potential (ordinary potential energy) $\Pi(q_i, t)$ we take generalised potential $V(q_i, \dot{q}_i, t)$ in terms of which the generalised force Q_i can be expressed by the formula

$$Q_i = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i}$$

Then the general equation of Lagrange's may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i}$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}\right\} - \frac{\partial}{\partial q_i}(T-V) = 0$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_i}(T-V)\right\} - \frac{\partial}{\partial q_i}(T-V) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where $L = T - V$

$V = V(q_i, \dot{q}_i, t)$

Expression for Kinetic energy :

Let the dynamical system consists of N-particles with masses m_j and position vectors $r_j = r_j(q_1, q_2, ---, q_n, t)$ where $q_1, q_2, ---, q_n$ are generalised

Co-ordinates.

Let T be the total kinetic energy then

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{r}_\nu^2$$

we have $r_\nu = r_\nu(q_1, q_2, \cdots, q_n, t)$

$$\Rightarrow \dot{r}_\nu = \sum_{i=1}^{n} \frac{\partial r_\nu}{\partial q_i}\dot{q}_i + \frac{\partial r_\nu}{\partial t}$$

$$\Rightarrow \dot{r}_\nu^2 = \left[\sum_{i=1}^{n} \frac{\partial r_\nu}{\partial q_i}\dot{q}_i + \frac{\partial r_\nu}{\partial t}\right]^2$$

$$= \left[\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial q_k}\dot{q}_i\dot{q}_k + 2\sum_{i=1}^{n}\frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial t}\dot{q}_i + \left(\frac{\partial r_\nu}{\partial t}\right)^2\right]$$

$$\therefore T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial q_k}\dot{q}_i\dot{q}_k + \sum_{\nu=1}^{N} m_\nu \sum_{i=1}^{n}\frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial t}\dot{q}_i + \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\frac{\partial r_\nu}{\partial t}\right)^2$$

$$\Rightarrow T = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\sum_{\nu=1}^{N} m_\nu \frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial q_k}\right]\dot{q}_i\dot{q}_k + \sum_{i=1}^{n}\left[\sum_{\nu=1}^{N} m_\nu \frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial t}\right]\dot{q}_i + \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\frac{\partial r_\nu}{\partial t}\right)^2$$

$$\Rightarrow T = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\dot{q}_i\dot{q}_k + \sum_{i=1}^{n} a_i\dot{q}_i + a_0$$

where $a_{ik} = \sum_{\nu=1}^{N} m_\nu \frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial q_k}$

$a_i = \sum_{\nu=1}^{N} m_\nu \frac{\partial r_\nu}{\partial q_i}\frac{\partial r_\nu}{\partial t}$ and $a_0 = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\frac{\partial r_\nu}{\partial t}\right)^2$

$$\Rightarrow T = T_2 + T_1 + T_0$$

Where $T_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \dot{q}_i \dot{q}_k$

$$T_1 = \sum_{i=1}^{n} a_i \dot{q}_i \text{ and } T_0 = a_0$$

It is obvious that T_2 is a homogeneous function of generalised velocities of second degree. T_1 is a homogeneous function of generalised velocities of degree one. If the system is scleronomic then $T_1 = T_0 = 0$. In this case K.E. $T = T_2$

Note: If $f = f(q_1, q_2, ---, q_n)$ be a homogeneous function of n variables in m degree then by Euler's equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial q_i} q_i = mf$$

$$\therefore \sum_{i=1}^{n} \frac{\partial T_2}{\partial \dot{q}_i} \dot{q}_i = 2T_2$$

and $$\sum_{i=1}^{n} \frac{\partial T_1}{\partial \dot{q}_i} \dot{q}_i = T_1$$